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SAMSO ltr 17 Jan 1972

AD815093

# Multiple Scattering and the Method of Rytov

**APRIL 1967**

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April 1967

Prepared for

BALLISTIC SYSTEMS AND SPACE SYSTEMS DIVISIONS
AIR FORCE SYSTEMS COMMAND
LOS ANGELES AIR FORCE STATION
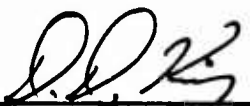Los Angeles, California

# FOREWORD

This report is published by the Aerospace Corporation, El Segundo, California, under Air Force Contract No. AF 04(695)-1001.

This report, which documents research carried out during February 1967, was submitted on 2 May 1967, for review and approval, to Captain Ronald J. Starbuck, SSTRT.

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# ABSTRACT

Rytov's method (method of smooth perturbations) has been used extensively to solve problems involving wave propagation in weakly inhomogeneous media. Critics of the method have questioned the closure procedure employed in the Rytov theory and have expressed doubt as to the range of validity of the solution. Some of the criticisms have claimed the method fails because it does not treat multiple scattering and that it is limited in applicability to precisely the same region as the Born single scattering approximation. The criticisms are shown herein to be at fault. A multiple scattering interpretation of the Rytov solution is given, thus lending support to those proponents of Rytov who claimed an extended range of applicability relative to the Born approximation.

# MULTIPLE SCATTERING AND THE METHOD OF RYTOV

The classical approach to the solution of problems involving wave propagation in weakly inhomogeneous random media has been via perturbation methods.[1-3] In general, only a first-order solution (Born or single scattering approximation) has been tractable and useful. This solution is necessarily limited to propagation over short distances by the requirement that perturbations be small. To overcome this limitation, Rytov[4,5] rewrote the inhomogeneous wave equation in the field variable in terms of the logarithm of the variable. He then obtained the usual inhomogeneous wave equation in the new variable by a questionable closure procedure.[6,7] Results obtained by Rytov's method have been widely used by workers in radio and optical propagation. Unfortunately, however, Rytov's closure procedure appears never to have been rigorously justified.

DeWolf[8] has shown that the Rytov approximation is equivalent to the Born and geometrical optics solutions in regions where the latter methods are applicable. He suggests a far greater range of applicability including regions where multiple scattering effects are important. Brown[9] and Taylor,[7,10] on the other hand, have taken the position that the Rytov approximation is valid only in regions where the Born approximation holds. Brown,[11] in addition, refers to the Rytov solution as a single scattering approximation. It is the purpose of this note to point out that (1) in a sense, the Rytov method does treat multiple scattering, and (2) on a proper interpretation, the Rytov solution is not restricted as claimed by Brown and Taylor.

As a starting point for the discussion, consider the scalar Helmholtz equation in one of the field variables

$$[\nabla^2 + k_0^2 n^2(\underline{r})] A(\underline{r}) = 0 \quad ,$$

where $A(\underline{r})$ is the field variable, $k_0$ is the wave number, and $n(\underline{r})$ is the randomly varying index of refraction. Let $n(\underline{r}) = 1 + N(\underline{r})$. Then we find

$$(\nabla^2 + k_0^2) A(\underline{r}) = -2k_0^2 N(\underline{r}) A(\underline{r}) - k_0^2 N^2(\underline{r}) A(\underline{r}) \quad . \tag{1}$$

We now let

$$A(\underline{r}) = \sum_{j=0}^{\infty} A_j(\underline{r}) \quad , \tag{2}$$

where $A_0(\underline{r})$ is the solution of

$$(\nabla^2 + k_0^2) A_0(\underline{r}) = 0 \quad . \tag{3}$$

Combining (1), (2), and (3) yields

$$(\nabla^2 + k_0^2) \sum_{j=1}^{\infty} A_j(\underline{r}) = [-2k_0^2 N(\underline{r}) - k_0^2 N^2(\underline{r})] \sum_{n=0}^{\infty} A_j(\underline{r}) \quad . \tag{4}$$

Let

$$(\nabla^2 + k_0^2) A_1(\underline{r}) = -2k_0^2 N(\underline{r}) A_0(\underline{r}) \quad . \tag{5}$$

The first-order perturbation equation (5) is recognized as the well-known inhomogeneous wave equation whose solution (Born or single scattering approximation) is

$$A_1(\underline{r}) = 2k_0^2 \int_V dv' G(\underline{r}, \underline{r}') N(\underline{r}') A_0(\underline{r}') \qquad , \qquad (6)$$

where $G(\underline{r}, \underline{r}')$ is the Green's function for the operator $(\nabla^2 + k_0^2)$. To obtain the higher-order scattering terms, we subtract (5) from (4) to obtain

$$(\nabla^2 + k_0^2) \sum_{j=2}^{\infty} A_j(\underline{r}) = -2k_0^2 N(\underline{r}) \sum_{j=1}^{\infty} A_j(\underline{r}) - k_0^2 N^2(\underline{r}) \sum_{j=0}^{\infty} A_j(\underline{r}) \qquad .$$

Now for $n \geq 2$, let

$$(\nabla^2 + k_0^2) A_j(\underline{r}) = -2k_0^2 N(\underline{r}) A_{j-1}(\underline{r}) - k_0^2 N^2(\underline{r}) A_{j-2}(\underline{r}) \qquad . \qquad (7)$$

Since, for the solution of interest

$$A_{j-1}(\underline{r}) / A_{j-2}(\underline{r}) \gg N(\underline{r}) \qquad ,$$

the second term on the right of (7) may be neglected. With this simplification and the definition

$$\rho_j(\underline{r}) = A_j(\underline{r}) / A_0(\underline{r})$$

the solution of (7) may be written as

$$\rho_j(\underline{r}) = \frac{A_j(\underline{r})}{A_0(\underline{r})} = 2k_0^2 \int_V dv'\ G(\underline{r}, \underline{r}')N(\underline{r}')\ \rho_{j-1}(\underline{r}')\frac{A_0(\underline{r}')}{A_0(\underline{r})} \qquad . \qquad (8)$$

On combining (3), (5), and (8), we obtain the complete perturbation solution, placing multiple scattering terms in evidence:

$$A(\underline{r}) = A_0(\underline{r}) \sum_{j=0}^{\infty} \rho_j(\underline{r})$$

$$= A_0(\underline{r})\,[1 + \rho_1(\underline{r}) + \rho_2(\underline{r}) + \dots ] \qquad . \qquad (9)$$

It proves interesting to write similarly the Rytov solution as a power series in orders of the refractivity N and to make a comparison with (9). The Rytov solution applies to the equation

$$\nabla^2 \psi(\underline{r}) + [\nabla \psi(\underline{r})]^2 + k_0^2 [1 + N(\underline{r})]^2 = 0 \qquad (10)$$

obtained by setting $A(\underline{r}) = e^{\psi(\underline{r})}$ in (1). Rytov set $\psi(\underline{r}) = \psi_0(\underline{r}) + \psi_1(\underline{r})$ and defined $\psi_0(\underline{r})$ as the solution of (10) for $N(\underline{r}) = 0$. He then obtained

$$\nabla^2 \psi_1(\underline{r}) + \nabla \psi_1(\underline{r}) \cdot [2\nabla \psi_0(\underline{r}) + \nabla \psi_1(\underline{r})] + 2k_0^2 N(\underline{r}) + k^2 N^2(\underline{r}) = 0$$

and effected closure by neglecting the terms in $[\nabla \psi_1(\underline{r})]^2$ and $N^2(\underline{r})$. The resulting solution for $\psi_1(\underline{r})$ is given[5] by (8) with j = 1. Thus, we obtain

-4-

$$A(\underline{r}) = \exp[\psi_0(\underline{r}) + \psi_1(\underline{r})] = A_0(\underline{r}) \exp[\rho_1(\underline{r})]$$

$$= A_0(\underline{r})[1 + \rho_1(\underline{r}) + \rho_1^2(\underline{r})/2 + \dots \rho_1^j(\underline{r})/n! + \dots]$$

$$= A_0(\underline{r})[1 + \sum_{n=1}^{\infty} \rho_j{}'(\underline{r})] \qquad , \tag{11}$$

where

$$\rho_j{}'(\underline{r}) = 2k_0^2 \int_v dv'\ G(\underline{r}, \underline{r}')N(\underline{r}') \frac{A_0(\underline{r}')}{A_0(\underline{r})} \frac{\rho_1^{j-1}(\underline{r})}{j!} \qquad . \tag{12}$$

A multiple scattering interpretation of the Rytov solution is evident upon comparison of (11) and (12) with (8) and (9). It is noted that zero and first-order scattering terms of the two series are identical. The second-order scattering term in the Rytov method, in the context of the successive perturbation analysis, is obtained by use of the approximation that the first-order scattered field everywhere in the scattering region is given by the constant $\rho_1(\underline{r})/2$, which is half of the first-order scattered field at the observation point $\underline{r}$. This is evidently a reasonable constant approximation to the scattered field since fluctuations due to scattering increase monotonically with penetration of the medium. Similarly, the jth-order scattering term is obtained by use of the approximation that the (j - 1)th scattered field is uniform in the scattering region and given by $\rho_1^{j-1}(\underline{r})/j!$.

The comparison of (9) and (11) reveals that the Rytov solution is valid everywhere in the region of validity of the Born approximation $[\rho(\underline{r}) \ll 1]$. More than that, it suggests a wider range of validity as was claimed by Chernov and Tatarski.[4, 5, 8]

Brown[7] has calculated a measure of the error in $\psi_1(\underline{r})$ occasioned by Rytov's closure procedure. He concludes that the error is small only when the Born approximation is valid. However, as Fried[12] has pointed out, Brown treats the solution for $\psi_1(\underline{r})$ in (11) as complete whereas it is more properly only a fluctuation. The latter interpretation is necessary to satisfy conservation of energy.[8, 13, 14] Thus, we find

$$\langle \exp[\psi_1(\underline{r}) + \psi_1^*(\underline{r})] \rangle = \langle \exp[2\chi(\underline{r})] \rangle$$

$$= \exp\{2[\langle \chi^2(\underline{r}) \rangle - \langle \chi(\underline{r}) \rangle^2 + \langle \chi(\underline{r}) \rangle]\}$$

$$= \exp[2(\sigma^2 + m)]$$

$$= 1 \quad , \tag{13}$$

where $\chi(\underline{r}) = \mathrm{Re}\{\psi_1(\underline{r})\}$, $\sigma^2$ is the variance, and m is the mean of $\chi(\underline{r})$. We have used the result that $\chi(\underline{r})$ is a Gaussian random variable[15] in the development of (16). But it is apparent from (6) that $\langle \chi(\underline{r}) \rangle = 0$ and that $\langle \chi^2(\underline{r}) \rangle \neq 0$. Hence, to satisfy (13), we interpret $\psi_1(\underline{r})$ as a fluctuation about a mean value $-\langle \chi^2(\underline{r}) \rangle$. It is to be noted that the solution of (10) contains an arbitrary constant. In establishing the energy balance, we are merely applying the appropriate boundary condition. Thus, the proper form of the Rytov solution is

$$A(\underline{r}) = A_0(\underline{r}) \exp[-\langle \chi^2(\underline{r}) \rangle + \psi_1(\underline{r})] \quad . \tag{14}$$

The multiple scattering interpretation made in connection with (11) applies
to (14), but now each scattered term must be reduced by $\exp[-\langle \chi^2(\underline{r}) \rangle]$.
Brown's conclusions, which are appropriate to (11), cannot be said to apply
to (14).

Taylor notes that (9) satisfies energy conservation to $O(N^2)$, i.e.,
when

$$A(\underline{r}) = A_0(\underline{r})[1 + \rho_1(\underline{r}) + \langle \rho_2(\underline{r}) \rangle] \qquad , \qquad (15)$$

and that a similar result is not obtained with the Rytov solution as given by
(11). He then concludes that $\psi_1(\underline{r})$ must be small if the energy balance is to
be approximately maintained. Interestingly, however, when the terms of
the multiple scattering series representation of (14) corresponding to those
in (15) are used, energy is also conserved. Thus, we find

$$[1 - 2\langle \chi^2(\underline{r}) \rangle] \langle |1 + \psi_1(\underline{r}) + \psi_1^2(\underline{r})/2|^2 \rangle$$

$$\approx [1 - 2\langle \chi^2(\underline{r}) \rangle][1 + \langle |\psi_1(\underline{r})|^2 \rangle + \langle \psi_1^2(\underline{r}) + \psi_1^{*2}(\underline{r}) \rangle/2]$$

$$\approx [1 - 2\langle \chi^2(\underline{r}) \rangle][1 + 2\langle \chi^2(\underline{r}) \rangle]$$

$$\approx 1$$

if terms of $O(N^4)$ and higher are neglected.

Since the arguments of Brown and Taylor do not apply to (14), it does not follow that the range of validity of the Rytov solution is that of the Born approximation. The claims of a greater range of validity made by the early proponents of Rytov seem reasonable and, indeed, are supported by recent experimental evidence.[16] The precise limitations remain inadequately explored.

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## DOCUMENT CONTROL DATA - R&D

*(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)*

| 1. ORIGINATING ACTIVITY (Corporate author) | 2a. REPORT SECURITY CLASSIFICATION |
|---|---|
| Aerospace Corporation<br>El Segundo, California | Unclassified |
| | 2b. GROUP |

**3. REPORT TITLE**

MULTIPLE SCATTERING AND THE METHOD OF RYTOV

**4. DESCRIPTIVE NOTES** *(Type of report and inclusive dates)*

**5. AUTHOR(S)** *(Last name, first name, initial)*

Heidbreder, G. R.

| 6. REPORT DATE | 7a. TOTAL NO. OF PAGES | 7b. NO. OF REFS |
|---|---|---|
| April 1967 | 13 | 16 |

| 8a. CONTRACT OR GRANT NO. | 9a. ORIGINATOR'S REPORT NUMBER(S) |
|---|---|
| AF 04(695)-1001 | TR-1001(2230-46)-6 |
| b. PROJECT NO. | |
| c. | 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report) |
| d. | SSD-TR-67-72 |

**10. AVAILABILITY/LIMITATION NOTICES**

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| 11. SUPPLEMENTARY NOTES | 12. SPONSORING MILITARY ACTIVITY |
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Rytov's method (method of smooth perturbations) has been used extensively to solve problems involving wave propagation in weakly inhomogeneous media. Critics of the method have questioned the closure procedure employed in the Rytov theory and have expressed doubt as to the range of validity of the solution. Some of the criticisms have claimed the method fails because it does not treat multiple scattering and that it is limited in applicability to precisely the same region as the Born single scattering approximation. The criticisms are shown herein to be at fault. A multiple scattering interpretation of the Rytov solution is given, thus lending support to those proponents of Rytov who claimed an extended range of applicability relative to the Born approximation.

14. KEY WORDS

Rytov Method
Atmospheric Propagation
Propagation in Random Media

Abstract (Continued)